Chapter 18

Performance of Feedback Systems

18.1 Introduction

It is now time to turn to issues of **performance**. As noted in earlier chapters, performance specifications typically involve the *closed-loop* relations between the exogenous inputs w and the regulated outputs z. These relationships are typically captured through the use of the signal and system norms. The *analysis* of a given controlled system usually involves evaluating the appropriate norms. The *synthesis* of a controller is a harder problem, as it involves picking a feedback compensator K for which the closed-loop performance specifications are attained.

We begin our discussions with the single-input, single-output (SISO) case, and then move on to study multi-input, multi-output (MIMO) extensions. Much of what we present for the SISO case actually echoes what is done in "classical feedback control", although our perspective is somewhat more modern (or neo-classical or post-modern or ...!).

18.2 SISO Loop Shaping

The Classical Viewpoint

The standard "servo" or *tracking* configuration of classical feedback control is shown in Figure 18.1. In this arrangement, the controller K is fed by an error signal e, which is the difference between a reference r and the measured output y of the plant P. The measurement is perhaps corrupted by noise n. The output of the controller is the input u to the plant. In addition, external disturbances may drive the plant, and are represented here via the signal d added in at the output of the plant. In a typical classical control design, the compensator K would be picked as the lowest-order system that ensures the following:

1. the closed-loop system is *stable*;

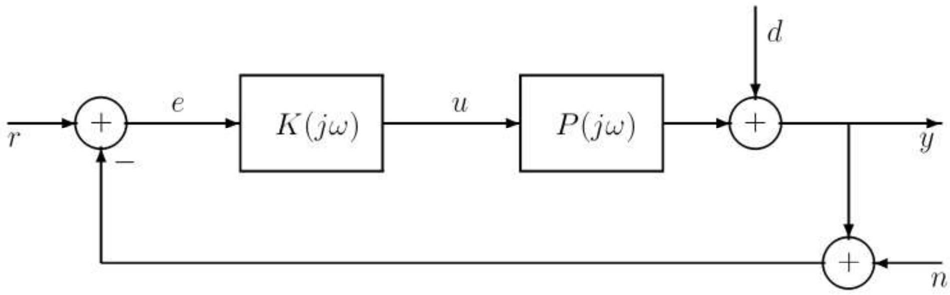

Figure 18.1: Standard feedback configuration with noise, disturbance, and reference inputs.

2. the **loop gain** $P(j\omega)K(j\omega)$ has *large* magnitude at frequencies (low frequencies, typically) where the power of the plant disturbance d or reference input r is concentrated;

3. the loop gain has *small* magnitude at frequencies (high frequencies, typically) where the power of the measurement noise n is concentrated.

The need for the first requirement is clear. The origins of the second and third requirements will be explained below. In order to simultaneously attain all three objectives, it is most convenient to have a criterion for closed-loop stability that is stated in terms of the (open-loop) loop gain, and this is provided by the *Nyquist stability criterion.*

The reasons for the second and third requirements above lie in the *sensitivities* of the closed-loop system to plant disturbances, reference signals, and measurement noise. Let S denote the transfer function that maps a disturbance d to the output y in the closed-loop system. This S is termed the (output) **sensitivity function**, and for the arrangement in Figure 18.1 it is given by

$$S = (1 + PK)^{-1} \quad . \tag{18.1}$$

Speaking informally for the moment, if $|P(j\omega)K(j\omega)|$ is large at frequencies where (in some sense) the power of d is concentrated, then $|S(j\omega)|$ will be small there, so the effect of the disturbance on the output will be attenuated. Since plant disturbances are typically concentrated around the low end of the frequency spectrum, one would want $|P(j\omega)K(j\omega)|$ to be large at low frequencies. Thus, *disturbance rejection* is a key motivation behind classical control's low-frequency specification on the loop gain.

Note that (in the SISO case) S is also the transfer function from r to e. If we want y to track r with good accuracy, then we want a small response of the error signal e to the driving signal r. This again leads us to ask for $|S(j\omega)|$ to be small — or equivalently for $|P(j\omega)K(j\omega)|$ to be large — at frequencies where the power of the reference signal r is concentrated. Fortunately, in many (if not most) control applications, the reference signal is slowly varying, so this requirement again reduces to asking for $|P(j\omega)K(j\omega)|$ to be large at low frequencies. Thus, *tracking accuracy* is another motivatoin behind classical control's low-frequency specification on the loop gain.

In contrast, the motivation behind classical control's high frequency specification is *noise rejection*. Let T denote the transfer function that maps the noise input n to the output y. Given the arrangement in Figure 18.1,

$$T = PK(1+PK)^{-1} . \tag{18.2}$$

This T is termed the **complementary sensitivity function**, because

$$T + S = 1 . \tag{18.3}$$

Note that T is also the transfer function from r to y. If $|P(j\omega)K(j\omega)|$ is small at frequencies where the power in n is concentrated, then $|T(j\omega)|$ will be small there, so the effect of the noise on the output will be attenuated. Measurement noise tends to occur at higher frequencies, so to minimize its effects on the output, we typically specify that $|P(j\omega)K(j\omega)|$ be small at high frequencies. This constraint fortunately does not conflict with the low frequency constraints imposed above by typical d and r. Also, the constraint is well matched to the inevitable fact that the gain of physical systems will eventually fall off with frequency.

The picture of the control design task that emerges from the above discussion is the following: Given the plant P, one typically needs to pick the compensator K so as to obtain a loop gain magnitude $|P(j\omega)K(j\omega)|$ that is large at low frequencies, "rolls off" to low values at high frequencies, and varies in such a way that the Nyquist stability criterion is satisfied. [For the special case of open loop stable plants and compensators, the stability condition can be stated in alternative forms that are easy to check using Bode plots rather than Nyquist plots, and this can be more convenient. The standard rule of thumb focuses on the roll off around the *crossover frequency* ω_c, defined as the frequency where the loop gain magnitude is unity; this frequency is a crude measure of closed loop bandwidth. The specification is that the roll off of the loop gain magnitude around ω_c should be no steeper than -20dB/decade. Furthermore, ω_c should be picked below frequencies where the loop gain is significantly affected by any right half plane zeros of the loop transfer function PK; this provides an initial indication that right half plane zeros can limit the attainable closed loop performance.]

A Modern Viewpoint

The challenge now is to translate the above classical control design approach into something more precise and systematic, and more likely to have a natural MIMO extension. The following example points the way, and makes free use of the signal and system norms that we defined in Lectures 11 and 12.

Example 18.1 (SISO Disturbance Rejection and Weighted Sensitivity)

We have already seen that the expression relating y to d in the SISO feedback configuration depicted in Figure 18.1 is

$$y = (1+PK)^{-1}d . \tag{18.4}$$

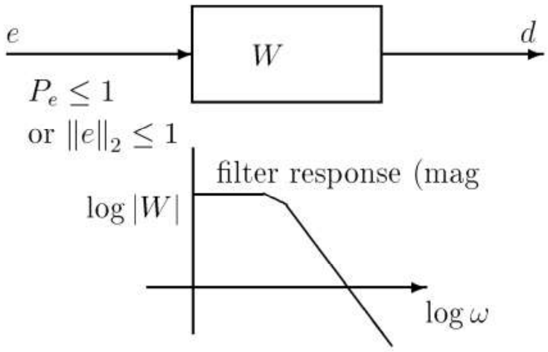

Figure 18.2: Representing the plant disturbance d as the output of a shaping filter W whose input e is an arbitrary bounded energy or bounded power signal, or possibly white noise.

Typically, d has frequency content concentrated in the low-frequency range. In order to get the requisite frequency characteristic, one might model d as the output of a *shaping filter* with transfer function W, as shown in Figure 18.2, with the input e of the filter being an arbitrary bounded energy or bounded power disturbance (or, in the stochastic setting, white noise). Thus e has no spectral "coloring", and all the coloring of d is embodied in the frequency response of W.

For the rest of this example, let us focus on the bounded energy or bounded power models for e. Suppose our goal now is to choose K to minimize the effect of the disturbance d on the output y. From Lectures 11 and 12, and given our model for d, we know that this is equivalent to minimizing the $\mathcal{H}_\infty$-gain of the transfer function from e to y, because in the case of a bounded power e this gain is the attainable or "tight" bound on the ratio of rms values at the output and input,

$$\frac{\rho_y}{\rho_e} \leq \|(1 + P(j\omega)K(j\omega))^{-1}W(j\omega)\|_\infty \quad ,$$

while in the case of a bounded energy e we again have the tight bound

$$\frac{\|y\|_2}{\|e\|_2} \leq \|(1 + P(j\omega)K(j\omega))^{-1}W(j\omega)\|_\infty \quad .$$

In terms of the sensitivity function,

$$S(j\omega) = (1 + P(j\omega)K(j\omega))^{-1} \quad ,$$

the task is to pick K to minimize the $\mathcal{H}_\infty$ norm $\|S(j\omega)W(j\omega)\|_\infty$.

If

$$\|S(j\omega)W(j\omega)\|_\infty \leq \gamma \, , \tag{18.5}$$

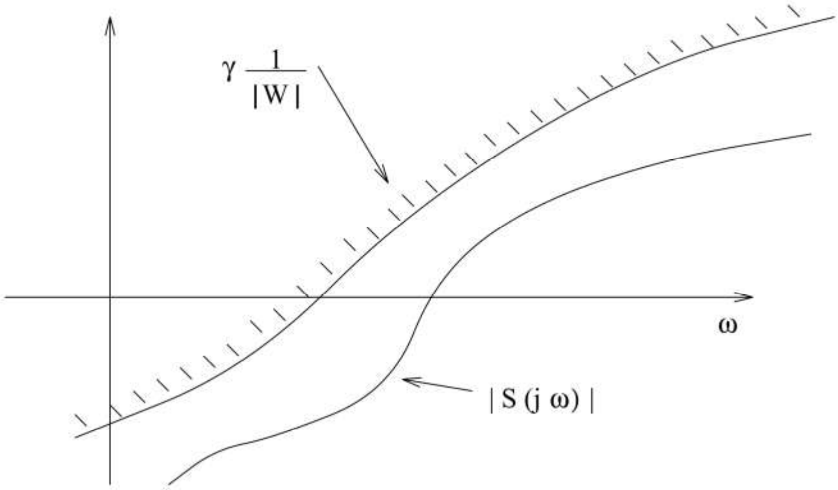

Figure 18.3: Graphical interpretation of the sensitivity function being bounded by a scaled reciprocal of the weighting filter frequency response.

then

$$|S(j\omega)|\,|W(j\omega)| \leq \gamma\ ,\ \ \forall\omega\ . \tag{18.6}$$

This implies that

$$|S(j\omega)| \leq \gamma\frac{1}{|W(j\omega)|}\quad , \tag{18.7}$$

which tells us that the sensitivity function is bounded by a scaled reciprocal of the weighting filter. A graphical representation of this bound is shown in Figure 18.3. From Figure 18.3 we can see that the value γ and the filter $W(j\omega)$ give us a clear picture of the constraint on the sensitivity function. This allows one to more systematically design a controller, since we directly get the closed loop characteristics. Note also that with the Q-parametrization of K, the sensitivity function S is affine in Q, and this form is much easier to work with than the fractional form that S takes as a function of K.

The major benefit of the formulation in the above example is that a MIMO version of it is quite immediate, as we see in the next section.

18.3 MIMO Loop Shaping

Let us now revisit the above example in the MIMO setting. The example will require the following facts about singular values, so we ask you to confirm these facts for yourself before proceeding:

1. $\sigma_{max}(AB) \leq \sigma_{max}(A)\sigma_{max}(B)$, and

2. If $\sigma_{max}(CD) < 1$ then $\sigma_{max}(C) < \dfrac{1}{\sigma_{min}(D)}$ assuming D is invertible.

The first statement follows from the fact that σ_{max} is the induced 2-norm, and therefore submultiplicative. To prove the second, apply the first with $A = CD$ and $B = D^{-1}$.)

Example 18.2 (MIMO Disturbance Rejection and Weighted Sensitivity)

The set-up and formulation for the MIMO case are the same as in the SISO example, with the obvious replacements of SISO subsystems by MIMO subsystems. One again arrives at the equation (18.5). However, the inference from this equation in the MIMO case is no longer (18.6) and (18.7), but rather

$$\sigma_{max}\left[(I + P(j\omega)K(j\omega))^{-1}\right] \leq \gamma \, \frac{1}{\sigma_{min}\left[W(j\omega)\right]} \quad .$$

This leads us to the singular value plot shown in Figure 18.4, which is the natural extension of the plot we had in the SISO example.

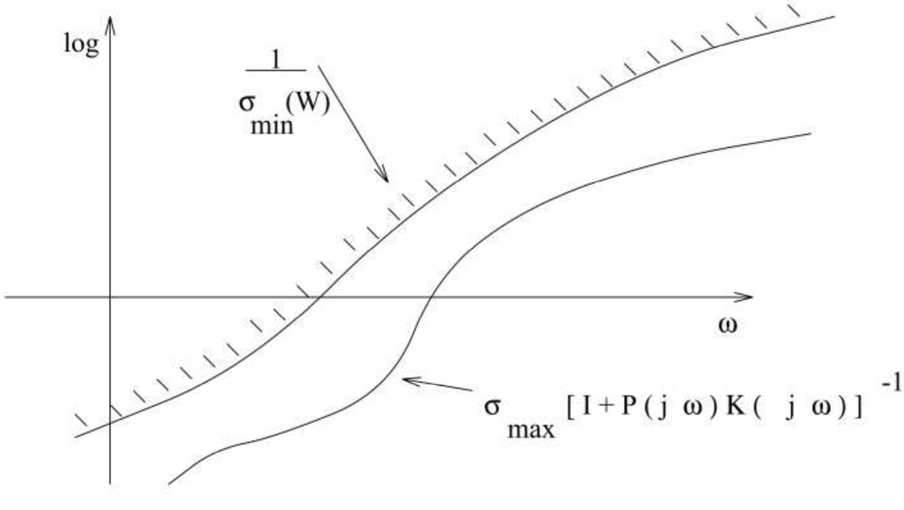

Figure 18.4: Singular value plot for a MIMO system.

With the insight provided by the above example, we can formulate a variety of MIMO performance problems in terms of appropriate weighting operators. Alternatively, having seen what sorts of modifications of the SISO statements are needed for the MIMO case, we can actually describe various MIMO control tasks in a language that is closer to that of classical SISO control, and this is what we do in the rest of this lecture. We shall return to the explicit use of weighting functions in later lectures.

Typical Closed-Loop Perfomance Constraints

Typically in control systems the disturbances d have frequency content that is concentrated in the low-frequency range. Therefore, in order to attenuate the effects of disturbances on the output, we require that $\sigma_{max}(S(j\omega))$ be small in the range of frequencies where the disturbances are active, say $0 \leq \omega \leq \omega_{sy}$. On the other hand, typically the noise input n has frequency content that is concentrated in the high-frequency range. Therefore, in order to attenuate the effect of n on the output we require that $\sigma_{max}(T(j\omega))$ be small over a frequency range of the form $\omega \geq \omega_{ty}$. The controller K should also enable the closed-loop system to track reference inputs r that are typically concentrated in the low frequency range, for example in the interval $0 \leq \omega \leq \omega_r$. This objective requires that $T(j\omega) \approx I$ for all ω in the interval $0 \leq \omega \leq \omega_r$. This requirement can be restated as

$$\begin{aligned}\sigma_{max}(T(j\omega)) &\approx 1 \\ \sigma_{min}(T(j\omega)) &\approx 1,\end{aligned}$$

in the frequency range $0 \leq \omega \leq \omega_r$.

The control signals must also generally be kept as small as possible in the presence of both disturbances d and measurement noise n. It is easy to see that

$$u = (I + KP)^{-1}Kr - (I + KP)^{-1}K(d + n) \; .$$

Therefore, in order to keep the control signal small, we must make sure that

$$\sigma_{max}\left((I + K(j\omega)P(j\omega))^{-1}K(j\omega)\right)$$

remains small in the frequency range where disturbances and/or measurement errors are effective. We can summarize these design requirements in the following table:

Design Requirement	Closed-Loop Condition	Frequency Range
Sensitivity to Disturbances	$\sigma_{max}\left((I + P(j\omega)K(j\omega))^{-1}\right) \approx 0$	Low frequency $0 \leq \omega \leq \omega_{sy}$
Noise Propagation Attenuation	$\sigma_{max}\left((I + P(j\omega)K(j\omega))^{-1}P(j\omega)K(j\omega)\right) \approx 0$	High Frequency $\omega \geq \omega_{ty}$
Tracking of Reference Signals	$\sigma_{max}\left((I + P(j\omega)K(j\omega))^{-1}P(j\omega)K(j\omega)\right) \approx 1$ $\sigma_{min}\left((I + P(j\omega)K(j\omega))^{-1}P(j\omega)K(j\omega)\right) \approx 1$	Low frequency $0 \leq \omega \leq \omega_r$
Low Control Energy	$\sigma_{max}\left((I + K(j\omega)P(j\omega))^{-1}K(j\omega)\right) \approx 0$	Frequencies where d and n are dominant

Translation to Open-Loop Constraints

Now let us relate the closed-loop requirements that are summarized in the preceding table to open-loop conditions, i.e., conditions on the singular values of the loop gain operator

PK. The first design requirement is that $\sigma_{max}\left((I+PK)^{-1}\right)$ be small in the frequency range $0 \leq \omega \leq \omega_{sy}$. The relation

$$\sigma_{max}\left((I+P(j\omega)K(j\omega))^{-1}\right) = \frac{1}{\sigma_{min}(I+P(j\omega)K(j\omega))}$$

implies that if $\sigma_{min}(P(j\omega)K(j\omega)) >> 1$ then

$$\sigma_{max}\left((I+P(j\omega)K(j\omega))^{-1}\right) \approx \frac{1}{\sigma_{min}(P(j\omega)K(j\omega))}\ . \tag{18.8}$$

Therefore, if $\sigma_{min}(P(j\omega)K(j\omega)) >> 1$ for all ω in the interval $[0, \omega_{sy}]$, then $\sigma_{max}\left((I+P(j\omega)K(j\omega))^{-1}\right)$ will be small in that interval.

For noise attenuation, consider

$$\begin{aligned}
\sigma_{max}\left(T(j\omega)\right) &= \sigma_{max}\left(I-(I+P(j\omega)K(j\omega))^{-1}\right) \\
&= \sigma_{max}\left(\left(I+(P(j\omega)K(j\omega))^{-1}\right)^{-1}\right) \\
&= \frac{1}{\sigma_{min}\left(I+(P(j\omega)K(j\omega))^{-1}\right)}.
\end{aligned}$$

Therefore, for the frequency range $\omega \geq \omega_{ty}$ we require that $\sigma_{min}\left(I+(P(j\omega)K(j\omega))^{-1}\right)$ be as large as possible. This can be guaranteed if we make $\sigma_{min}\left((P(j\omega)K(j\omega))^{-1}\right)$ as large as possible or equivalently by making $\sigma_{max}\left(P(j\omega)K(j\omega)\right)$ as small as possible.

The tracking objective can be achieved if we ensure that

$$\begin{aligned}
\sigma_{max}\left((I+P(j\omega)K(j\omega))^{-1}P(j\omega)K(j\omega)\right) &\approx 1 \\
\sigma_{min}\left((I+P(j\omega)K(j\omega))^{-1}P(j\omega)K(j\omega)\right) &\approx 1
\end{aligned}$$

over the frequency interval $[0, \omega_r]$. Since

$$I-(I+P(j\omega)K(j\omega))^{-1} = (I+P(j\omega)K(j\omega))^{-1}P(j\omega)K(j\omega)$$

the tracking objective can be achieved if we require $(I+P(j\omega)K(j\omega))^{-1}$ to be close to zero on the frequency range $[0, \omega_r]$, that is $\sigma_{max}\left((I+P(j\omega)K(j\omega))^{-1}\right)$ to be small in that interval. Equivalently, we may require $\sigma_{min}\left(I+P(j\omega)K(j\omega)\right)$ to be as large as possible on the interval $[0, \omega_r]$. This can be ensured if we require that $\sigma_{min}\left(P(j\omega)K(j\omega)\right)$ be as large as possible over the frequency range $[0, \omega_r]$.

The constraint of small control energy leads to the condition that $\sigma_{max}\left((I+K(j\omega))P(j\omega))^{-1}K(j\omega)\right)$ be made as small as possible. However, we have

$$\begin{aligned}
\sigma_{max}\left((I+K(j\omega)P(j\omega))^{-1}K(j\omega)\right) &\leq \sigma_{max}\left((I+K(j\omega)P(j\omega))^{-1}\right)\sigma_{max}(K(j\omega)) \\
&= \frac{\sigma_{max}(K(j\omega))}{\sigma_{min}\left(I+K(j\omega)P(j\omega)\right)}.
\end{aligned} \tag{18.9}$$

Note that

$$\begin{aligned}\sigma_{min}\left(I+K(j\omega)P(j\omega)\right) &\leq \sigma_{max}\left(I+K(j\omega)P(j\omega)\right)\\ &\leq 1+\sigma_{max}(P(j\omega))\sigma_{max}(K(j\omega))\end{aligned}$$

so

$$\begin{aligned}\frac{\sigma_{max}(K(j\omega))}{\sigma_{min}\left(I+K(j\omega)P(j\omega)\right)} &\geq \frac{\sigma_{max}(K(j\omega))}{1+\sigma_{max}(P(j\omega))\sigma_{max}(K(j\omega))}\\ &= \frac{1}{\frac{1}{\sigma_{max}(K(j\omega))}+\sigma_{max}(P(j\omega))}.\end{aligned}$$

Therefore, we can minimize the right hand side of equation 18.9 only if we make

$$\frac{1}{\sigma_{max}(K(j\omega))}+\sigma_{max}(P(j\omega))$$

large in the ranges of frequencies where d and/or n are dominant. For example, if $\sigma_{max}(P(j\omega))$ is small at a certain set of frequencies of interest then necessarily $\sigma_{max}(K(j\omega))$ must also be small on that set. Clearly this condition is not necessary or sufficient to make

$$\sigma_{max}\left((I+K(j\omega)P(j\omega))^{-1}K(j\omega)\right)$$

small. It only applies to the upper bound of $\sigma_{max}\left((I+K(j\omega)P(j\omega))^{-1}K(j\omega)\right)$, which is given by

$$\frac{\sigma_{max}(K(j\omega))}{\sigma_{min}\left(I+K(j\omega)P(j\omega)\right)}$$

and it is only necessary for the upper bound to be small.

The following table summarizes our discussion above on open-loop requirements

Design Requirement	Open-Loop Condition	Frequency Range
Sensitivity to Disturbances	$\sigma_{min}\left(P(j\omega)K(j\omega)\right)$ large	Low frequency $0 \leq \omega \leq \omega_{sy}$
Noise Propagation Attenuation	$\sigma_{max}\left(P(j\omega)K(j\omega)\right)$ small	High Frequency $\omega \geq \omega_{ty}$
Tracking of Reference Signals	$\sigma_{min}\left(P(j\omega)K(j\omega)\right)$ large	Low frequency $0 \leq \omega \leq \omega_r$
Low Control Energy	$\sigma_{max}\left(K(j\omega)\right)$ small	Frequencies where $\sigma_{max}\left(P(j\omega)\right)$ is not large enough

Figure 18.6 illustrates the open-loop conditions that we have formulated. Note that in this plot the minimum passband open-loop gain is bounded by $\sigma_{min}\left[P(j\omega)K(j\omega)\right]$, and the maximum stopband open loop gain bounded by $\sigma_{max}\left[P(j\omega)K(j\omega)\right]$.

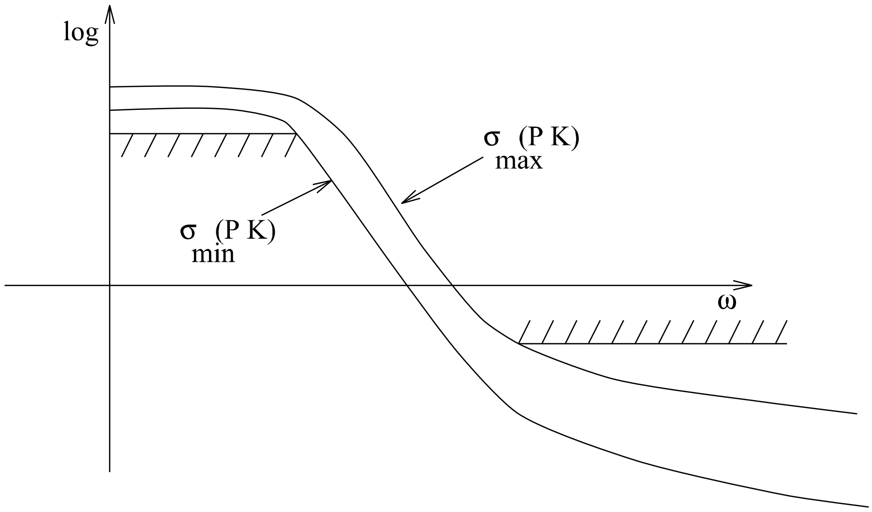

Figure 18.5: Singular value bounds for the open loop gain, $P(j\omega)K(j\omega)$.

18.4 Algebraic Constraints

In general we would like to design feedback controllers to attenuate both noise and disturbances at the output. We have examined SISO and MIMO conditions that guarantee rejection of low frequency disturbances as well as similar conditions for the rejection of high frequency noise. However, one might wonder if we can

1. minimize the influence of either noise or disturbances over all frequencies, and/or
2. minimize the influence of both noise and disturbances at the same frequency.

Let us begin this discussion by recalling the following:

- $S = (I + PK)^{-1}$ is the transfer function mapping disturbances to the output;
- $T = PK(I + PK)^{-1}$ is the transfer function mapping noise to the output.

As we mentioned earlier, in a control design it is usually desirable to make both S and T small. However, because of algebraic constraints, both goals are not simultaneously achievable at the same frequency. These constraints are as follows.

General Limitations

$S + T = I$ for all complex (Laplace domain) frequencies s. This is easily verified, since

$$\begin{aligned} S + T &= (I + PK)^{-1} + PK(I + PK)^{-1} \\ &= (I + PK)(I + PK)^{-1} \\ &= I . \end{aligned}$$

This result implies that if $\sigma_{max}[S(j\omega)]$ is small in some frequency range, $\sigma_{max}[T(j\omega)] \sim 1$. The converse is also true.

Fortunately, we rarely need to make both of these functions small in the same frequency region.

Limitations Due to RHP Zeros and Poles

Before we discuss these limitations, we quote the following fact from complex analysis:

> Let $H(s)$ be a stable, causal, linear time invariant continuous time system. The *maximum modulus principle* implies that
>
> $$\sigma_{max}[H(s)] \leq \sup_{\omega} \sigma_{max}[H(j\omega)] = \|H\|_{\infty} \quad \forall\, s \in \text{RHP}.$$

In other words, a stable function, which is analytic in the RHP, achieves its maximum value over the RHP when evaluated on the imaginary axis.

Using this result, we can arrive at relationships between poles and zeros of the plant P located in the RHP and limitations on performance (*e.g.*, disturbance and noise rejection).

SISO Systems: Disturbance Rejection

Consider the stable sensitivity function $S = (1 + PK)^{-1}$ for any stabilizing controller, K; then,

$$\begin{aligned} S(z_i) &= (1 + P(z_i)K(z_i))^{-1} = 1 \quad \text{for all RHP zeros } z_i \text{ of } P \\ S(p_i) &= (1 + P(p_i)K(p_i))^{-1} = 0 \quad \text{for all RHP poles } p_i \text{ of } P. \end{aligned}$$

Since the $\mathcal{H}_{\infty}$ norm bounds the gain of a system over all frequencies,

$$1 = |S(z_i)| \leq \|S\|_{\infty}.$$

This means that we cannot uniformly attenuate disturbances over the entire frequency range if there are zeros in the RHP.

SISO Systems: Noise Rejection

Since the transfer function relating a noise input to the output is $T = PK(1 + PK)^{-1}$, an argument for T similar to S can be made, but with the roles of poles and zeros interchanged. In this case, RHP poles of the plant restrict us from uniformly attenuating noise over the entire frequency range.

MIMO Systems: Disturbance Rejection

Suppose P has a transmission zero at $\tilde{z} \in$ RHP with left input zero direction η^*. Then $\eta^* P(\tilde{z})K(\tilde{z}) = 0$, and thus

$$\eta^*(I + P(\tilde{z})K(\tilde{z}))^{-1} = \eta^* \quad .$$

Stated equivalently,

$$\eta^* S(\tilde{z}) = \eta^* \quad . \tag{18.10}$$

Also, taking the conjugate transpose of both sides,

$$S^*(\tilde{z})\eta = \eta \quad . \tag{18.11}$$

We then multiply the expressions in (18.10) and (18.11), obtaining

$$\eta^* S(\tilde{z})S^*(\tilde{z})\eta = \eta^*\eta \quad ,$$

which can be alternately written as

$$\frac{\eta^* S(\tilde{z})S^*(\tilde{z})\eta}{\eta^*\eta} = 1 \quad . \tag{18.12}$$

Applying the maximum modulus principle (*i.e.*, $\max_{s \in \mathrm{RHP}} \sigma_{max}[S(s)]$ occurs on the imaginary axis) and observing that the left hand side of (18.12) is less than or equal to $\sigma^2_{max}[S(\tilde{z})]$, we conclude that

$$\|S\|^2_\infty \geq \frac{\eta^* S(\tilde{z})S^*(\tilde{z})\eta}{\eta^*\eta} = 1 \quad .$$

Thus, the conclusion regarding disturbance rejection for MIMO systems is the same as the conclusion we reached for SISO systems. Namely, RHP zeros make disturbance attenuation over all frequencies impossible.

18.5 Analytic Constraints: The "Waterbed Effect"

One performance limitation of LTI SISO Feedback systems (these systems have rational sensitivity transfer functions), is known as the *waterbed effect.* Loosely speaking, when one designs a controller to "push" the sensitivity function in a particular direction, another part of the sensitivity function necessarily "pulls" back in the opposite direction. This effect is due to a property of analytic functions $f(s)$ as stated by Cauchy's theroem. In words, this theorem says that the line integral of an analytic function around any simple closed contour C in a region $\mathbf{R}$ is zero, *i.e.*,

$$\int_C f(s)ds = 0.$$

for every contour C in $\mathbf{R}$.

A proof of this theorem will not be shown here but can be found in standard complex analysis textbooks. One consequence of this theorem is the following integral constraint (known as *Bode's Integral*) on the rational sensitivity transfer function $S(jw)$:

$$\int_0^\infty ln\ S(jw\ dw = \sum_i \pi Re(p_i),$$

where $\sum_i \pi Re(p_i)$, is the sum over the unstable open loop poles (poles of $P(jw)K(jw)$). This result holds for all closed loop systems as long as the product PK has relative degree two. The result implies that making $S(jw)$ small at almost all frequencies (a common performance objective) is impossible since the integrated value of $ln\ S(jw)$ over all frequencies must be constant. This constant is zero for open loop stable systems (PK stable) and positive otherwise. Therefore, lowering the sensitivity function in one range of frequencies, increases the same function in another range hence the name "waterbed effect." Figure 18.5 below illustrates this phenomenon.

Figure 18.6: Water bed Effect

Constraints on Singular Value Plots

From what we have seen already, it is clear that singular value plots over all frequencies are the MIMO system analogs of Bode plots. The following fact establishes some simple bounds involving singular values of S and T:

Fact 18.5.1 *If $S = (I + PK)^{-1}$ and $T = (I + PK)^{-1}PK$ then the following hold*

$$1 - \sigma_{max}(S) \leq \sigma_{max}(T) \leq 1 + \sigma_{max}(S)$$

and

$$1 - \sigma_{max}(T) \leq \sigma_{max}(S) \leq 1 + \sigma_{max}(T).$$

Proof: Since $S + T = I$ then clearly

$$\sigma_{max}(T) = \sigma_{max}(I - S) \leq \sigma_{max}(I) + \sigma_{max}(S),$$

and therefore $\sigma_{max}(T) \leq 1 + \sigma_{max}(S)$. For any element $x \in \mathbb{C}^n$ with $\|x\|_2 = 1$ we have

$$\begin{aligned} x - Sx &= Tx \\ \|x\|_2 - \|Sx\|_2 &\leq \|x - Sx\|_2 = \|Tx\|_2 \\ 1 - \|Sx\|_2 &\leq \sigma_{max}(T) \\ 1 - \sigma_{max}(S) &\leq \sigma_{max}(T). \end{aligned}$$

Combining this relation with $\sigma_{max}(T) \leq 1 + \sigma_{max}(S)$, we obtain

$$1 - \sigma_{max}(S) \leq \sigma_{max}(T) \leq 1 + \sigma_{max}(S).$$

The other relation follows in exactly the same manner.